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On Nearly Basic Rough Sets and Their Medical Applications

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Abstract

The rapid advancement of Pawlak's rough set (PRS) theory has motivated researchers to develop enhanced methodologies for decision-making, including applications in medical diagnosis particularly in identifying disease risk factors. The primary objective of this study is to introduce new mathematical techniques based on basic rough sets (as defined by Abu-Gdairi et al., 2021), employing nearly open concepts to enhance precision. Specifically, we propose the framework of nearly basic rough sets, incorporating pre-, semi-, and $\gamma-$approximations, and present three distinct approximation methods. Their fundamental properties and interrelationships are systematically examined. The proposed approaches demonstrate superior accuracy compared with existing methods, as confirmed through both theoretical analysis and illustrative case studies. In particular, the application to COVID-19 diagnosis yielded an accuracy of 100% in the tested case. Moreover, we introduce a mathematical algorithm suitable for implementation in programming languages, thereby facilitating broader applications in medical research, economic modeling, and related theoretical domains.

Keywords: $\mathfrak{b}-$neighborhoods; $\mathfrak{b}-$rough sets; Nearly $\mathfrak{b}-$rough sets; COVID-19 variants.

1 Introduction

Recently, Pawlak's rough set (PRS) theory and its subsequent generalizations have gained significant attention in the fields of computer science and artificial intelligence (AI). First formulated by Pawlak [15] in 1982, this theory has proven effective in addressing problems of incomplete information. It is founded on the classification of objects through an equivalence relation, emphasizing the completeness of information regarding a given set. By utilizing approximation operators and measures of precision, the theory provides critical insights for decision-makers concerning the formation and scope of boundary regions. However, the rigid requirements inherent in equivalence relations impose notable constraints on the real-world applicability of Pawlak's foundational framework.

To address these constraints, several generalizations have been proposed through broader relational frameworks. Yao [21] first introduced such an approach in 1996, giving rise to numerous generalized rough set models based on distinct types of relations, including tolerance relations [20], similarity relations [2, 8], quasi-orders [16], and binary relations [3, 7].

Building upon the concepts of right and left neighborhoods, Abo Khadra et al. [12] introduced a topological perspective in 2007. They developed an innovative method to construct topologies directly from binary relations, thereby eliminating the need for a basis or subbasis. This advancement expanded the applicability of rough sets within a topological framework and broadened the accessibility of the theory beyond specialists. In this framework, a topology $\mathcal{T}$ on a universe $\mathcal{U}$ is defined as follows,

$$\mathcal{T} = \{\mathcal{M} \subseteq \mathbb{U} : \mho(\d{t}) \subseteq \mathcal{M},\ \forall \d{t} \in \mathcal{M}\}, \tag{1}$$

where $\mho(\d{t})$ denotes the neighborhood of each element $\d{t}$ in $\mathbb{U}$.

In 2014, Abd El-Monsef et al. [1] expanded on the work of Abo Khadra et al. by introducing the concept of a $j-$neighborhood space $(j-\boldsymbol{NS})$. This framework enhanced neighborhood-based rough set theory by incorporating novel types of neighborhoods constructed through intersections and unions of left and right neighborhoods, as originally outlined by Yao, together with the minimal neighborhoods proposed by Allam et al. (2005) [7]. This resulted in eight distinct types of neighborhoods, offering a flexible means to generalize Pawlak's model without imposing restrictive relational assumptions. The $j-\boldsymbol{NS}$ framework has since been applied to various topological structures within rough set theory.

The notion of "initial-neighborhoods" was introduced for the first time in 2021 by El-Sayed et al. [9], constructed from right neighborhoods according to the formula,

$$\mho_{\mathfrak{i}}(\d{g}) = \{\d{w} \in \mathbb{U} : \mho_{\mathfrak{r}}(\d{g}) \subseteq \mho_{\mathfrak{r}}(\d{w})\}. \tag{2}$$

Here, $\mho_{\mathfrak{r}}(\d{g})$ denotes a right neighborhood of $\d{g}$. Their study demonstrated the use of initial-neighborhoods in rough set theory and highlighted their potential in addressing practical challenges such as those posed by COVID-19 through generalized nano-topology.

Around the same time, Abu-Gdairi et al. [5] introduced the complementary concept of "basic-neighborhoods" defined as,

$$\mho_{\mathfrak{b}}(\d{g}) = \{\d{w} \in \mathbb{S} : \mho_{\mathfrak{r}}(\d{w}) \subseteq \mho_{\mathfrak{r}}(\d{g})\}. \tag{3}$$

The relationships between these neighborhood concepts were further clarified in [5], where it was shown that their intersection corresponds to the core neighborhood of $\d{g}$. This concept was first suggested in [11] and later extended in [22].

Topological structures thus encompass a wide array of concepts and results that have proven valuable across multiple domains [13, 23]. Many researchers have further explored these structures in different theoretical frameworks, including Increasing and Decreasing Rough Set Approximations via Nano-Ordered Topology [17] and Increasing and Decreasing Soft Rough Set Approximations [10], as well as the medical applications of tritopological approximation spaces [14], which may also extend to algebraic topology-such as in the study of topological rings and topological modules [6].

The COVID-19 pandemic, with its variants such as Alpha, Beta, Gamma, Delta, and Omicron, has severely strained healthcare systems due to their high transmissibility and mortality rates, while also causing significant social and psychological disruptions. Advanced methods [18, 19] are therefore essential for improving the prediction, diagnosis, and management of such variants. Building on this foundation, Abu-Gdairi and El-Bably [4] recently investigated topological applications in rough set models, applying and analyzing the notion of nearly open sets in the context of "initial-rough sets."

In the current manuscript, we extend these ideas by applying concepts of nearly open sets to "basic-rough sets." This research introduces novel methodologies grounded in topological principles-particularly those related to nearly open concepts-to refine rough set granulation and enhance their application in COVID-19 diagnosis. By developing near basic-rough sets, namely $\mathcal{P}_{\mathfrak{b}}-$rough sets, $\mathcal{S}_{\mathfrak{b}}-$rough sets, and $\gamma_{\mathfrak{b}}-$rough sets, this study directly constructs general rough sets from binary relations, thereby circumventing the need for induced topologies. Practical applications demonstrate the efficiency of these models in analyzing COVID-19 variants, proposing a precise technique for decision-makers in healthcare management.

First, we illustrate that the proposed techniques and those introduced by Abu-Gdairi and El-Bably [4] are incomparable. Furthermore, we provide several comparisons between the suggested methods and previous approaches, showing that our methods yield greater accuracy than existing techniques.

This paper emphasizes its main contributions through the following key aspects:

1. **Introduction of novel approaches:** Innovative methods for constructing topological basic rough sets are proposed, eliminating the reliance on generated topology. The concept of nearly basic approximations is introduced, offering enhanced precision compared to existing approaches, as demonstrated through illustrative examples and counterexamples.

2. **COVID-19 variant classification:** A rule-based classification framework is developed, leveraging these methodologies to accurately identify and predict COVID-19 variants. Applying these methods to real data highlights the effectiveness of nearly basic approximations in uncovering significant trends and supporting informed decision-making.

3. **Mathematical and practical significance:** This research presents a noteworthy advancement in mathematics, enhancing decision-making accuracy while being supported by a practical framework for identifying COVID-19 variants. It significantly contributes to saving resources and time for both patients and healthcare professionals.

4. **Comparative evaluation:** The newly introduced nearly basic approximations (pre-, semi-, and $\gamma-$) are rigorously compared with established methods, showcasing their distinct advantages and superior performance.

5. **Algorithm and broader framework:** An algorithm optimized for MATLAB implementation is proposed, together with a framework for extending these methods to various domains,

including medical and economic applications. The study also outlines potential future directions, such as addressing big data tasks and expanding the versatility of these methodologies.

2 Fundamental Principles

In this section, key concepts of rough set theory are unified, leveraging foundational works. Mathematically, a binary relation $\mathcal{R}$ on a non-empty set $\mathbb{U}$ is defined as a subset of the Cartesian product $\mathbb{U} \times \mathbb{U}$. For any $\mathfrak{s}, \mathfrak{t} \in \mathbb{U}$, the relation $\mathfrak{s}\mathcal{R}\mathfrak{t}$ holds if and only if $(\mathfrak{s}, \mathfrak{t}) \in \mathcal{R}$. In this study, $\mathbb{U}$ is assumed to be finite.

The classification of different types of binary relations is formalized as follows:

Definition 2.1. [21] *Let $\mathcal{R}$ be a binary relation on $\mathbb{U}$. Then, $\mathcal{R}$ can be characterized by the following properties:*

1. ***Serial:*** *For all $\mathfrak{s} \in \mathbb{U}$, there exists a $\mathfrak{t} \in \mathbb{U}$ such that $\mathfrak{s}\mathcal{R}\mathfrak{t}$.*
2. ***Reflexive:*** *For every $\mathfrak{s} \in \mathbb{U}$, the relation $\mathfrak{s}\mathcal{R}\mathfrak{s}$ is satisfied.*
3. ***Symmetric:*** *If $\mathfrak{s}\mathcal{R}\mathfrak{t}$, then $\mathfrak{t}\mathcal{R}\mathfrak{s}$, for all $\mathfrak{s}, \mathfrak{t} \in \mathbb{U}$.*
4. ***Transitive:*** *For all $\mathfrak{s}, \mathfrak{q}, \mathfrak{t} \in \mathbb{U}$, $\mathfrak{s}\mathcal{R}\mathfrak{q}$ and $\mathfrak{q}\mathcal{R}\mathfrak{t} \Rightarrow \mathfrak{s}\mathcal{R}\mathfrak{t}$.*
5. ***Preorder (or quasi-order):*** *A relation $\mathcal{R}$ satisfies conditions (2) and (4).*
6. ***Equivalence:*** *The relation $\mathcal{R}$ meets the criteria specified in (2)–(4).*

Definition 2.2. *Consider the binary relation $\mathcal{R}$ on a finite set $\mathbb{U}$. For each $\mathfrak{s} \in \mathbb{U}$:*

1. *$\mho_{\mathfrak{r}}(\mathfrak{s}) = \{\mathfrak{t} \in \mathbb{U} : \mathfrak{s}\mathcal{R}\mathfrak{t}\}$, called the right neighborhood of $\mathfrak{s}$ [21].*
2. *$\mho_{\mathfrak{m}}(\mathfrak{s}) = \cup_{\mathfrak{s} \in \mho_{\mathfrak{r}}(\mathfrak{t})} \mho_{\mathfrak{r}}(\mathfrak{t})$, called the maximal neighborhood of $\mathfrak{s}$ [7].*
3. *$\mho_{\mathfrak{b}}(\mathfrak{s}) = \{\mathfrak{t} \in \mathbb{U} : \mho_{\mathfrak{r}}(\mathfrak{t}) \subseteq \mho_{\mathfrak{r}}(\mathfrak{s})\}$, called the basic-neighborhood of $\mathfrak{s}$ [5].*

Using the above neighborhoods, many rough approximations can be formed, as explained below.

Definition 2.3. *The lower and upper approximations, boundary region, and the accuracy measures of approximations, for every $\mathcal{M} \subseteq \mathbb{U}$ and $\forall j \in \{\mathfrak{r}, \mathfrak{m}, \mathfrak{b}, \mathfrak{i}\}$, are defined respectively by:*

$$\mathcal{L}_j(\mathcal{M}) = \{\mathfrak{s} \in \mathbb{U} : \mho_j(\mathfrak{s}) \subseteq \mathcal{M}\}, \qquad \mathcal{U}_j(\mathcal{M}) = \{\mathfrak{s} \in \mathbb{U} : \mho_j(\mathfrak{s}) \cap \mathcal{M} \neq \emptyset\},$$

$$\mathfrak{B}_j(\mathcal{M}) = \mathcal{U}_j(\mathcal{M}) - \mathcal{L}_j(\mathcal{M}), \qquad \mathcal{A}_j(\mathcal{M}) = \frac{|\mathcal{L}_j(\mathcal{M})|}{|\mathcal{U}_j(\mathcal{M})|}, \quad \text{where} \quad |\mathcal{U}_j(\mathcal{M})| \neq 0.$$

It is necessary to point out that:

1. When $j = \mathfrak{r}$, Definition 2.3 corresponds to Yao's approach [21].
2. When $j = \mathfrak{m}$, Definition 2.3 corresponds to the approach of Dai et al. [8].
3. When $j = \mathfrak{b}$, Definition 2.3 corresponds to the approach of Abu-Gdairi et al. [5].

4. When $j = \mathfrak{i}$, Definition 2.3 corresponds to El-Sayed et al. [9] model.

Abd El-Monsef et al. [1] proposed an important approach for constructing a general topology derived from a binary relation, formalized in the following theorem.

Theorem 2.1. [1] *Given a relation $\mathcal{R}$, a topology on $\mathbb{U}$ can be defined as,*

$$\mathcal{T}_{\mathfrak{r}} = \{\mathcal{M} \subseteq \mathbb{U} : \mho_{\mathfrak{r}}(\mathfrak{s}) \subseteq \mathcal{M},\ \forall \mathfrak{s} \in \mathcal{M}\}.$$

Definition 2.4. [1] *Let $\mathcal{T}_{\mathfrak{r}}$ denote the topology on $\mathbb{U}$ induced by the relation $\mathcal{R}$. The sets in $\mathcal{T}_{\mathfrak{r}}$ are called $\mathfrak{r}-$open sets, and their complements are referred to as $\mathfrak{r}-$closed sets. The family of all $\mathfrak{r}-$closed sets is denoted by,*

$$\mathcal{C}_{\mathfrak{r}} = \{\mathcal{H} \subseteq \mathbb{U} : \mathcal{H}^c \in \mathcal{T}_{\mathfrak{r}}\}.$$

Building on this topological framework, Abd El-Monsef et al. [1] introduced rough approximations, defined as follows:

Definition 2.5. [1] *For a topology $\mathcal{T}_{\mathfrak{r}}$ on $\mathbb{U}$ induced by a relation $\mathcal{R}$, the rough approximations of a set $\mathcal{M} \subseteq \mathbb{U}$ are given by,*

$$\begin{aligned} \underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M}) &= \bigcup \{\mathcal{G} \in \mathcal{T}_{\mathfrak{r}} : \mathcal{G} \subseteq \mathcal{M}\}, \\ \overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M}) &= \bigcap \{\mathcal{H} \in \mathcal{C}_{\mathfrak{r}} : \mathcal{M} \subseteq \mathcal{H}\}, \\ \mathfrak{B}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{M}) &= \overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M}) \setminus \underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M}), \\ \text{and} \quad \mathcal{A}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{M}) &= \frac{|\underline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M})|}{|\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M})|}, \quad \text{provided that} \quad |\overline{\mathcal{T}}_{\mathfrak{r}}(\mathcal{M})| \neq 0. \end{aligned}$$

3 Rough Set Approximations Based on Nearly-Open Concepts and Basic Rough Sets

In this section, we introduce new generalizations and refinements of basic rough sets [5], providing highly accurate alternative approximation operators. The proposed methods are constructed using topological structures derived from nearly-open concepts (specifically pre-open, semi-open, and $\gamma-$open sets), without relying on conventional formal topologies.

These approaches offer a more intuitive framework for researchers who are not specialized in topology, while simultaneously expanding the potential applications of topological methods across a wide range of scientific disciplines. We investigate the fundamental properties of the proposed operators and establish their interrelations, supported by formal proofs and illustrative examples.

Definition 3.1. *Let $\mathcal{R}$ be a binary relation on $\mathbb{U}$. For each $\mathcal{M} \subseteq \mathbb{U}$, the following approximation operators are defined:*

1. *Pre-basic approximations:*

$$\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})), \qquad \overline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M} \cup \mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M})).$$

2. *Semi-basic approximations:*

$$\underline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M} \cap \mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M})), \qquad \overline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M} \cup \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})).$$

3. *$\gamma-$basic approximations:*

$$\underline{\gamma}_{\mathfrak{b}}(\mathcal{M}) = \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \cup \underline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M}), \qquad \overline{\gamma}_{\mathfrak{b}}(\mathcal{M}) = \overline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \cap \overline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M}).$$

From these operators, we define the nearly basic boundary region and the nearly basic accuracy of approximations, respectively as,

$$\mathfrak{B}^{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) = \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) - \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}),$$
$$\mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M}) = \frac{|\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})|}{|\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})|},$$

where $\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \neq \varnothing$ and $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$.

The operators defined above are collectively referred to as nearly basic approximations, or simply $\mathcal{N}_{\mathfrak{b}}-$approximations. It follows immediately that the accuracy of approximation satisfies,

$$0 \leq \mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M}) \leq 1.$$

When $\mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M}) = 1$, the set $\mathcal{M}$ is said to be $\mathcal{N}_{\mathfrak{b}}-$definable (or $\mathcal{N}_{\mathfrak{b}}-$exact). Otherwise, it is categorized as a $\mathcal{N}_{\mathfrak{b}}-$rough set.

The following proposition establishes several fundamental properties of the $\mathcal{N}_{\mathfrak{b}}-$approximations.

Proposition 3.1. *Let $\mathcal{R}$ be a binary relation on $\mathbb{U}$. For all $\mathcal{M}, \mathcal{F} \subseteq \mathbb{U}$ and for each $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, the following statements hold:*

(1) $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})$.

(2) $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathbb{U}) = \overline{\mathcal{N}}_{\mathfrak{b}}(\mathbb{U}) = \mathbb{U}$ *and* $\underline{\mathcal{N}}_{\mathfrak{b}}(\varnothing) = \overline{\mathcal{N}}_{\mathfrak{b}}(\varnothing) = \varnothing$.

(3) *If* $\mathcal{M} \subseteq \mathcal{F}$, *then* $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(4) *If* $\mathcal{M} \subseteq \mathcal{F}$, *then* $\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(5) $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M} \cap \mathcal{F}) = \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \cap \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(6) $\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M} \cup \mathcal{F}) = \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \cup \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(7) $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M} \cup \mathcal{F}) \supseteq \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \cup \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(8) $\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M} \cap \mathcal{F}) \subseteq \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \cap \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{F})$.

(9) $\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) = \left[\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}^c)\right]^c$, *where* $\mathcal{M}^c$ *denotes the complement of* $\mathcal{M}$.

(10) $\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) = \left[\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}^c)\right]^c$.

(11) $\underline{\mathcal{N}}_{\mathfrak{b}}\left(\underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})\right) = \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})$.

(12) $\overline{\mathcal{N}}_{\mathfrak{b}}\left(\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})\right) = \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M})$.

Proof. We provide the proof for the case $\mathcal{N} = \mathcal{P}$; the arguments for $\mathcal{S}$ and γ follow analogously:

(1) These follow directly from Definition 3.1.

(2) These follow directly from Definition 3.1.

(3) Suppose $\mathcal{M} \subseteq \mathcal{F}$. Then, by [5], we have $\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{F})$ and $\mathcal{U}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{F})$. Consequently,

$$\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M} \cap \mathcal{L}_{\mathfrak{b}}\big(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})\big) \subseteq \mathcal{F} \cap \mathcal{L}_{\mathfrak{b}}\big(\mathcal{U}_{\mathfrak{b}}(\mathcal{F})\big) = \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{F}).$$

(4) The proof proceeds in the same manner as in (3).

(5) Since $\mathcal{M} \cap \mathcal{F} \subseteq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{F} \subseteq \mathcal{F}$, applying (3) gives,

$$\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M} \cap \mathcal{F}) \subseteq \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \cap \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{F}).$$

Conversely, let $s \in \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \cap \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{F})$. Then,

$$s \in \mathcal{M} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})) \quad \text{and} \quad s \in \mathcal{F} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{F})).$$

By [5], it follows that,

$$s \in (\mathcal{M} \cap \mathcal{F}) \cap \mathcal{L}_{\mathfrak{b}}\big(\mathcal{U}_{\mathfrak{b}}(\mathcal{M} \cap \mathcal{F})\big) = \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M} \cap \mathcal{F}).$$

Hence, equality holds.

(6) This can be established by an argument similar to (5).

(7) These follow directly from the monotonicity established in (3) and (4).

(8) These follow directly from the monotonicity established in (3) and (4).

(9) We compute

$$\big[\overline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}^c)\big]^c = \big[\mathcal{M}^c \cup \mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M}^c))\big]^c = \mathcal{M} \cap \big[\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M}^c))\big]^c.$$

By [5], this reduces to

$$\mathcal{M} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})) = \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}).$$

(10) The argument is analogous to (9).

(11) From (1), we know that,

$$\underline{\mathcal{P}}_{\mathfrak{b}}\big(\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M})\big) \subseteq \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}).$$

Now, suppose $s \notin \underline{\mathcal{P}}_{\mathfrak{b}}(\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}))$. Then,

$$s \notin \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \quad \text{or} \quad s \notin \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}))).$$

This implies that $\mho_{\mathfrak{b}}(s) \cap \mathcal{U}_{\mathfrak{b}}(\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M})) = \varnothing$, and therefore $s \notin \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M})$. Hence, equality holds.

(12) The proof is entirely analogous to that of (11).

□

The subsequent results present a comparative analysis among various types of nearly-basic approximations.

Proposition 3.2. *Suppose that $\mathcal{R}$ be binary relation on $\mathbb{U}$ and $\mathcal{M} \subseteq \mathbb{U}$, then the following statements hold:*

(1) $\underline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \underline{\gamma}_{\mathfrak{b}}(\mathcal{M})$.

(2) $\underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \underline{\gamma}_{\mathfrak{b}}(\mathcal{M})$.

(3) $\overline{\gamma}_{\mathfrak{b}}(\mathcal{M}) \subseteq \overline{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M})$.

(4) $\overline{\gamma}_{\mathfrak{b}}(\mathcal{M}) \subseteq \overline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M})$.

Proof. By Definition 3.1, the proof is obvious. □

Corollary 3.1. *Let $\mathcal{R}$ be a binary relation on $\mathbb{U}$, and let $\mathcal{M} \subseteq \mathbb{U}$. Then, the following statements hold:*

(1) $\mathfrak{B}^{\gamma}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathfrak{B}^{\mathcal{S}}_{\mathfrak{b}}(\mathcal{M})$.

(2) $\mathfrak{B}^{\gamma}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathfrak{B}^{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M})$.

(3) $\mathcal{A}_{\mathcal{S}_{\mathfrak{b}}}(\mathcal{M}) \leq \mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$.

(4) $\mathcal{A}_{\mathcal{P}_{\mathfrak{b}}}(\mathcal{M}) \leq \mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$.

(5) *If $\mathcal{M}$ is $\mathcal{S}_{\mathfrak{b}}-$exact, then $\mathcal{M}$ is $\gamma_{\mathfrak{b}}-$exact and if $\mathcal{M}$ is $\mathcal{P}_{\mathfrak{b}}-$exact, then $\mathcal{M}$ is $\gamma_{\mathfrak{b}}-$exact.*

Remark 3.1.

(i) *The opposite of the above consequences is normally not correct.*

(ii) *$\mathcal{S}_{\mathfrak{b}}-$approximations and $\mathcal{P}_{\mathfrak{b}}-$approximations are mutually exclusive.*

The following example (Example 3.1) illustrates Remark 3.1 by presenting a comparative analysis of the different types of nearly basic-approximations. This example highlights the distinctions among the proposed approaches and demonstrates their effectiveness in approximating sets within the framework of nearly basic-approximations, thereby providing a concrete illustration of the concepts discussed in the remark.

Example 3.1. *Let the binary relation,*

$$\mathcal{R} = \Big\{(\d{g},\d{g}),\ (\d{g},\d{h}),\ (\d{g},\d{y}),\ (\d{g},\d{z}),\ (\d{h},\d{g}),\ (\d{y},\d{y}),\ (\d{y},\d{z}),\ (\d{z},\d{y}),\ (\d{z},\d{z})\Big\},$$

be defined on the universe $\mathbb{U} = \{\d{g},\d{h},\d{y},\d{z}\}$. Then, we obtain

$$\mho_{\mathfrak{b}}(\d{g}) = \mathbb{U}, \quad \mho_{\mathfrak{b}}(\d{h}) = \{\d{h}\}, \quad \mho_{\mathfrak{b}}(\d{y}) = \{\d{y},\d{z}\}, \quad \mho_{\mathfrak{b}}(\d{z}) = \{\d{y},\d{z}\}.$$

To demonstrate the validity of Remark 3.1 and to clarify the differences among the proposed nearly basic-approximations, we compare several measures of basic accuracy $\mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M})$ for all subsets $\mathcal{M} \subseteq \mathbb{U}$, where $\mathcal{N} = \{\mathcal{P},\mathcal{S},\gamma\}$. This comparative analysis is summarized in Table 1.

Table 1: Comparative analysis of different types of nearly-basic accuracy measures.

$\mathcal{M} \subseteq \mathbb{U}$	$\mathcal{A}_{\mathcal{P}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{S}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$
{g̣}	0	0	0
{ḥ}	$\frac{1}{2}$	1	1
{ỵ}	1	0	1
{ẓ}	1	0	1
{g̣, ḥ}	$\frac{1}{2}$	1	1
{g̣, ỵ}	$\frac{1}{2}$	0	$\frac{1}{2}$
{g̣, ẓ}	$\frac{1}{2}$	0	$\frac{1}{2}$
{ḥ, ỵ}	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{3}$
{ḥ, ẓ}	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{3}$
{ỵ, ẓ}	$\frac{2}{3}$	1	1
{g̣, ḥ, ỵ}	1	$\frac{1}{2}$	1
{g̣, ḥ, ẓ}	1	$\frac{1}{2}$	1
{g̣, ỵ, ẓ}	$\frac{2}{3}$	1	1
{ḥ, ỵ, ẓ}	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{U}$	1	1	1

Remark 3.2. *Proposition 3.2 emphasizes that the most accurate technique for approximating rough sets is through the use of $\gamma_{\mathfrak{b}}-$approximations. Moreover, Figure 1 illustrates the interrelations among the different categories of approximate sets.*

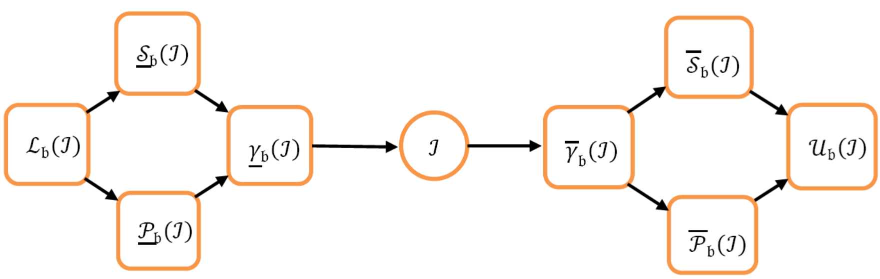

Figure 1: Interconnections between various categories of nearly-basic approximations and basic approximations.

4 Comparative Analysis of Nearly $\mathfrak{b}-$Approximations and Prior Approaches

In this section, we conduct a series of comparative examinations between the proposed techniques and several preceding approaches, including those of Dai et al. [8], Yao [21], Abu Gadairi

[5], and Abd El-Monsef et al. [1]. In addition, proven theorems and counterexamples are provided to illustrate these comparative studies.

First, we demonstrate that the proposed methods (nearly $\mathfrak{b}-$approximations) and the recently introduced techniques of Abu-Gdairi and El-Bably [4] (nearly $\mathfrak{i}-$approximations) are non-comparable.

(a) A comparison of current methods (nearly $\mathfrak{b}-$approximations) with Abu-Gdairi and El-Bably [4] techniques (nearly $\mathfrak{i}-$approximations)

Recently, Abu-Gdairi and El-Bably [4] applied topological concepts within rough set models, specifically studying the notion of nearly open sets in the context of initial rough sets ($\mathfrak{i}-$approximations). In this section, we first demonstrate that basic rough sets and initial rough sets are non-comparable. Thus, we establish that nearly basic-rough sets and nearly initial-rough sets are also non-comparable. To begin, we compare the structures of initial-approximations and basic-approximations. The following example illustrates that $\mathfrak{b}-$approximations and $\mathfrak{i}-$approximations are not comparable.

Example 4.1. *Consider a relation $\mathcal{R}$ defined on the universal set $\mathbb{U} = \{\d{g}, \d{h}, \d{y}, \d{z}\}$ as,*

$$\mathcal{R} = \left\{(\d{g},\d{g}), (\d{g},\d{h}), (\d{h},\d{g}), (\d{h},\d{h}), (\d{y},\d{g}), (\d{z},\d{g}), (\d{z},\d{z})\right\}.$$

The derived right-neighborhood sets are

$$\mho_{\mathfrak{r}}(\d{g}) = \mho_{\mathfrak{r}}(\d{h}) = \{\d{g},\d{h}\}, \quad \mho_{\mathfrak{r}}(\d{y}) = \{\d{g}\}, \quad \mho_{\mathfrak{r}}(\d{z}) = \{\d{g},\d{z}\}.$$

Subsequently, the basic-neighborhoods are determined as,

$$\mho_{\mathfrak{b}}(\d{g}) = \mho_{\mathfrak{b}}(\d{h}) = \{\d{g},\d{h},\d{y}\}, \quad \mho_{\mathfrak{b}}(\d{y}) = \{\d{y}\}, \quad \mho_{\mathfrak{b}}(\d{z}) = \{\d{y},\d{z}\}.$$

Additionally, the initial-neighborhoods are

$$\mho_{\mathfrak{i}}(\d{g}) = \mho_{\mathfrak{i}}(\d{h}) = \{\d{g},\d{h}\}, \quad \mho_{\mathfrak{i}}(\d{y}) = \mathbb{U}, \quad \mho_{\mathfrak{i}}(\d{z}) = \{\d{z}\}.$$

It is evident that the three types of neighborhoods defined above are not comparable.

Now, consider $\mathcal{M} = \{\d{g}, \d{h}, \d{y}\}$ and $\mathcal{K} = \{\d{g}, \d{h}, \d{z}\}$. By using Definition 2.3, the $\mathfrak{b}-$approximations of $\mathcal{M}$ are

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M}, \quad \mathcal{U}_{\mathfrak{b}}(\mathcal{M}) = \mathbb{U},$$

while the $\mathfrak{i}-$approximations of $\mathcal{M}$ are

$$\mathcal{L}_{\mathfrak{i}}(\mathcal{M}) = \{\d{g}, \d{h}\}, \quad \mathcal{U}_{\mathfrak{i}}(\mathcal{M}) = \mathcal{M}.$$

On the other hand, the $\mathfrak{b}-$approximations of $\mathcal{K}$ are

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{K}) = \varnothing, \quad \mathcal{U}_{\mathfrak{b}}(\mathcal{K}) = \mathcal{K},$$

while the $\mathfrak{i}-$approximations of $\mathcal{K}$ are

$$\mathcal{L}_{\mathfrak{i}}(\mathcal{K}) = \mathcal{K}, \quad \mathcal{U}_{\mathfrak{i}}(\mathcal{K}) = \mathbb{U}.$$

Remark 4.1. *We now illustrate the relationship between the nearly $\mathfrak{b}-$approximations and the nearly $\mathfrak{i}-$approximations. The following example (Example 4.2) shows that the nearly $\mathfrak{b}-$approximations and nearly $\mathfrak{i}-$approximations are non-comparable.*

Example 4.2. *Building on Example 4.1, we now illustrate Remark 4.1 in the context of γ_j-rough sets for each $j \in \{\mathfrak{b}, \mathfrak{i}\}$, with the other cases following similarly.*

Let $\mathcal{M} = \{\dot{g}\}$ and $\mathcal{K} = \{\dot{y}\}$. By Definition 2.3, the $\gamma_{\mathfrak{b}}-$approximations of $\mathcal{M}$ are

$$\underline{\gamma}_{\mathfrak{b}}(\mathcal{M}) = \varnothing, \quad \overline{\gamma}_{\mathfrak{b}}(\mathcal{M}) = \mathcal{M},$$

whereas the $\gamma_{\mathfrak{i}}-$approximations of $\mathcal{M}$ are

$$\underline{\gamma}_{\mathfrak{i}}(\mathcal{M}) = \overline{\gamma}_{\mathfrak{i}}(\mathcal{M}) = \mathcal{M}.$$

On the other hand, the $\gamma_{\mathfrak{b}}-$approximations of $\mathcal{K}$ are

$$\underline{\gamma}_{\mathfrak{b}}(\mathcal{K}) = \mathcal{K}, \quad \overline{\gamma}_{\mathfrak{b}}(\mathcal{K}) = \mathbb{U},$$

while the $\gamma_{\mathfrak{i}}-$approximations of $\mathcal{K}$ are

$$\underline{\gamma}_{\mathfrak{i}}(\mathcal{K}) = \varnothing, \quad \overline{\gamma}_{\mathfrak{i}}(\mathcal{K}) = \mathcal{K}.$$

(b) A comparison of current methods with the Abu-Gdairi [5] technique

Firstly, we will propose Lemma 4.1, which elucidates the connections between the methods proposed by Abu-Gdairi [5] and our suggested methods, which are nearly $\mathfrak{b}-$approximations.

Lemma 4.1. *If $\mathcal{R}$ is a binary relation on $\mathbb{U}$. Then, for every $\mathcal{M} \subseteq \mathbb{U}$, the following statements hold:*

(1) *The lower approximation of $\mathcal{M}$ based on the $\mathfrak{b}-$neighborhood is a subset of the lower approximation of the upper $\mathfrak{b}-$neighborhood of $\mathcal{M}$,*

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})).$$

(2) *The upper approximation of the lower $\mathfrak{b}-$neighborhood of $\mathcal{M}$ is contained within the upper $\mathfrak{b}-$neighborhood of $\mathcal{M}$,*

$$\mathcal{U}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M})) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{M}).$$

Proof. The item (1) will proved and the second follows in a similar manner. By the results in [5], we know that,

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M})) = \mathcal{L}_{\mathfrak{b}}(\mathcal{M}),$$

and

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{M}).$$

Thus,

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{L}_{\mathfrak{b}}(\mathcal{M})) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})),$$

which implies,

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})).$$

□

Theorem 4.1. *Given $\mathcal{R}$ is a binary relation on $\mathbb{U}$. For every subset $\mathcal{M} \subseteq \mathbb{U}$ and any $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, the following relations hold:*

(1) *The lower $\mathfrak{b}-$approximation of $\mathcal{M}$ is a subset of the lower approximation of $\mathcal{M}$ with respect to the $\mathfrak{b}-$neighborhood,*

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}).$$

(2) *The upper approximation of $\mathcal{M}$ with respect to the $\mathfrak{b}-$neighborhood is contained within the upper $\mathfrak{b}-$approximation of $\mathcal{M}$,*

$$\overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{M}).$$

Proof. The first statement will demonstrate for the case $\mathcal{N} = \mathcal{P}$, as the reasoning for the other cases is similar.

Suppose that $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{M})$. Then, there exists $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{M})$ such that $w \in \mho_{\mathfrak{b}}(w)$, which indicates $w \in \mathcal{M}$. By Lemma 4.1, we know that,

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M})).$$

Therefore, $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M}))$, and consequently,

$$w \in [\mathcal{M} \cap \mathcal{L}_{\mathfrak{b}}(\mathcal{U}_{\mathfrak{b}}(\mathcal{M}))] = \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}).$$

Hence, we conclude that,

$$\mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \underline{\mathcal{P}}_{\mathfrak{b}}(\mathcal{M}).$$

□

Corollary 4.1. *For a binary relation $\mathcal{R}$, $\mathcal{M} \subseteq \mathbb{U}$ and for every $\mathcal{M} \subseteq \mathbb{U}$ and any $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, the following results hold:*

(1) $\mathfrak{B}_{\mathfrak{b}}^{\mathcal{N}}(\mathcal{M}) \subseteq \mathfrak{B}_{\mathfrak{b}}(\mathcal{M})$.

(2) $\mathcal{A}_{\mathfrak{b}}(\mathcal{M}) \leq \mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M})$.

(3) *If $\mathcal{M}$ is an $\mathcal{N}_{\mathfrak{b}}-$exact set, it is $\mathfrak{b}-$exact.*

Remark 4.2. *According to Theorem 4.1 and Corollary 4.1, the techniques presented in Definitions 3.1 and 3.2 extend the approaches introduced by Abu-Gadairi [5]. In other words, the nearly $\mathfrak{b}-$approximations provide greater accuracy compared to those methods. Nevertheless, the converse of these results does not hold in general, as illustrated in Example 4.3.*

Example 4.3. *Drawing from Example 3.1, we contrast the outcomes derived from Abu-Gadairi et al.'s [5] method with those from the proposed $\mathcal{N}_{\mathfrak{b}}-$approximations, as outlined in Table 2.*

Table 2: Comparatives of accuracy measures: Abu-Gadairi et al.'s approach [5] versus the present method ($\mathcal{N}_{\mathfrak{b}}-$accuracy, for each $\mathcal{N} \in \{\mathcal{P},\ \mathcal{S},\ \gamma\}$).

$\mathcal{M} \subseteq \mathbb{U}$	Abu-Gdairi et al. [5]	Current methods		
	$\mathcal{A}_{\mathfrak{b}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{P}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{S}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$
$\{\d{g}\}$	0	0	0	0
$\{\d{h}\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\{\d{y}\}$	0	1	0	1
$\{\d{z}\}$	0	1	0	1
$\{\d{g}, \d{h}\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\{\d{g}, \d{y}\}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{\d{g}, \d{z}\}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{\d{h}, \d{y}\}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{3}$
$\{\d{h}, \d{z}\}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{3}$
$\{\d{y}, \d{z}\}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1
$\{\d{g}, \d{h}, \d{y}\}$	$\frac{1}{4}$	1	$\frac{1}{2}$	1
$\{\d{g}, \d{h}, \d{z}\}$	$\frac{1}{4}$	1	$\frac{1}{2}$	1
$\{\d{g}, \d{y}, \d{z}\}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1
$\{\d{h}, \d{y}, \d{z}\}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{U}$	1	1	1	1

(c) Comparing the nearly $\mathfrak{b}-$approximations to the method by Dai et al. [8]

For general binary relations, both $\mathfrak{b}-$rough approximations and the maximal rough approximations introduced by Dai et al. [8] are regarded as distinct constructs, as discussed in [5]. Accordingly, maximal approximations and $\mathcal{N}_{\mathfrak{b}}-$approximations maintain their individuality within this framework. It is noteworthy that the approximation approach proposed by Dai et al. departs from the fundamental principles of Pawlak's philosophy.

In contrast, the nearly $\mathfrak{b}-$approximations proposed herein generally satisfy all of Pawlak's axioms without imposing additional restrictions (see Section 3.1). Subsequent results and illustrative examples further clarify the relationships between the proposed methods and Dai et al.'s rough approximations, particularly in contexts involving general binary relations.

Lemma 4.2. [5] *Let $\mathcal{R}$ be a reflexive relation on $\mathbb{U}$, and let $\mathcal{M} \subseteq \mathbb{U}$. The following properties hold:*

1. $\mathcal{L}_{\mathfrak{m}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{m}}(\mathcal{M})$.
2. $\mathfrak{B}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathfrak{B}_{\mathfrak{m}}(\mathcal{M})$, *and* $\mathcal{A}_{\mathfrak{m}}(\mathcal{M}) \leq \mathcal{A}_{\mathfrak{b}}(\mathcal{M})$.

3. *If $\mathcal{M}$ is an $\mathfrak{m}$-exact set, then $\mathcal{M}$ is also $\mathfrak{b}-$exact.*

Theorem 4.2. *Let $\mathcal{R}$ be a reflexive relation on $\mathbb{U}$, and let $\mathcal{M} \subseteq \mathbb{U}$. For each nearly basic-approximation operator $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, the following properties hold:*

1. $\mathcal{L}_{\mathfrak{m}}(\mathcal{M}) \subseteq \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{m}}(\mathcal{M})$.
2. $\mathfrak{B}_{\mathfrak{b}}^{\mathcal{N}}(\mathcal{M}) \subseteq \mathfrak{B}_{\mathfrak{m}}(\mathcal{M})$, *and* $\mathcal{A}_{\mathfrak{m}}(\mathcal{M}) \leq \mathcal{A}_{\mathcal{N}\mathfrak{b}}(\mathcal{M})$.
3. *If $\mathcal{M}$ is an $\mathfrak{m}$-exact set, then $\mathcal{M}$ is also $\mathcal{N}_{\mathfrak{b}}$-exact.*

Proof. The result follows directly from Theorem 4.1 and Lemma 4.2. □

Remark 4.3. *Example 4.4 shows that the converse of the above theorem does not necessarily hold.*

Example 4.4. *Let the universe be $\mathbb{U} = \{g, h, y, z\}$, and consider the reflexive relation,*

$$\mathcal{R} = \{(g, g), (g, y), (h, h), (h, y), (y, y), (z, z), (h, g), (g, z), (y, z), (z, y)\}.$$

The corresponding $\mathfrak{b}-$neighborhoods are

$$\mho_{\mathfrak{b}}(g) = \{g, y, z\}, \quad \mho_{\mathfrak{b}}(h) = \{h\}, \quad \mho_{\mathfrak{b}}(y) = \{y, z\}, \quad \mho_{\mathfrak{b}}(z) = \{y, z\}.$$

Similarly, the $\mathfrak{m}-$neighborhoods are

$$\mho_{\mathfrak{m}}(g) = \mathbb{U}, \quad \mho_{\mathfrak{m}}(h) = \{g, h, y\}, \quad \mho_{\mathfrak{m}}(y) = \mathbb{U}, \quad \mho_{\mathfrak{m}}(z) = \{g, y, z\}.$$

Therefore, the $\mathfrak{m}-$accuracy approach [8] can be compared with the proposed $\mathcal{N}_{\mathfrak{b}}-$accuracy method, for every $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, as summarized in Table 3.

Table 3: Accuracy comparison between Dai et al. [8] and the proposed nearly $\mathfrak{b}-$approximations.

$\mathcal{M} \subseteq \mathbb{U}$	Dai et al. [8]	Proposed methods		
	$\mathcal{A}_{\mathfrak{m}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{P}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{S}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$
$\{g\}$	0	0	0	0
$\{h\}$	0	1	1	1
$\{y\}$	0	1	0	1
$\{z\}$	0	1	0	1
$\{g, h\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{g, y\}$	0	1	0	1
$\{g, z\}$	0	1	0	1
$\{h, y\}$	0	1	$\frac{1}{4}$	1
$\{h, z\}$	0	1	$\frac{1}{4}$	1
$\{y, z\}$	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\{g, h, y\}$	$\frac{1}{4}$	1	$\frac{1}{4}$	1
$\{g, h, z\}$	0	1	$\frac{1}{4}$	1
$\{g, y, z\}$	$\frac{1}{4}$	1	1	1
$\{h, y, z\}$	0	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{U}$	1	1	1	1

(d) Comparing the Yao [21] method with nearly $\mathfrak{b}-$approximations

For general binary relations, both $\mathfrak{b}-$rough approximations and Yao's rough sets [21] are regarded as independent approaches, as discussed in [5]. Consequently, Yao's approximations and nearly $\mathfrak{b}-$approximations retain their distinctiveness within this framework. It is important to note that Yao's models do not conform to the original axioms of Pawlak's philosophy, as illustrated in Example 2.2. In contrast, the proposed nearly $\mathfrak{b}-$approximations generally satisfy all of Pawlak's principles without imposing additional constraints.

The following results and examples clarify the relationships between the proposed methods and Yao's rough approximations, particularly in contexts involving binary relations.

Lemma 4.3. [5] *Let $\mathcal{R}$ be a reflexive relation on $\mathbb{U}$. Then, for all $w \in \mathbb{U}$, we have*

$$\mho_{\mathfrak{b}}(w) \subseteq \mho_{\mathfrak{r}}(w).$$

Theorem 4.3. *Suppose that $\mathcal{R}$ is a reflexive relation on $\mathbb{U}$, and let $\mathcal{M} \subseteq \mathbb{U}$. The following statements hold:*

1. $\mathcal{L}_{\mathfrak{r}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \mathcal{U}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{M})$.
2. $\mathfrak{B}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathfrak{B}_{\mathfrak{r}}(\mathcal{M})$, *and* $\mathcal{A}_{\mathfrak{r}}(\mathcal{M}) \leq \mathcal{A}_{\mathfrak{b}}(\mathcal{M})$.
3. *The $\mathfrak{r}-$exactness of $\mathcal{M}$ implies its $\mathfrak{b}-$exactness.*

Proof. We verify the first claim; the remaining results follow by a similar argument.
Assume $w \in \mathcal{L}_{\mathfrak{r}}(\mathcal{M})$, which implies $\mho_{\mathfrak{r}}(w) \subseteq \mathcal{M}$. By Lemma 4.3, we obtain $\mho_{\mathfrak{b}}(w) \subseteq \mathcal{M}$, hence $w \in \mathcal{L}_{\mathfrak{b}}(\mathcal{M})$. Therefore, $\mathcal{L}_{\mathfrak{r}}(\mathcal{M}) \subseteq \mathcal{L}_{\mathfrak{b}}(\mathcal{M})$. By analogous reasoning, $\mathcal{U}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{M})$. □

Note: Theorem 4.3 establishes a connection between Yao's technique ($\mathfrak{r}-$rough sets) and Abu-Gdairi's technique ($\mathfrak{b}-$rough sets) in the case where $\mathcal{R}$ is reflexive.

Theorem 4.4. *Let $\mathcal{R}$ be a reflexive relation on $\mathbb{U}$, and let $\mathcal{M} \subseteq \mathbb{U}$. For any $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$, the following hold:*

1. $\mathcal{L}_{\mathfrak{r}}(\mathcal{M}) \subseteq \underline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \overline{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathcal{U}_{\mathfrak{r}}(\mathcal{M})$.
2. $\mathfrak{B}^{\mathcal{N}}_{\mathfrak{b}}(\mathcal{M}) \subseteq \mathfrak{B}_{\mathfrak{r}}(\mathcal{M})$, *and* $\mathcal{A}_{\mathfrak{r}}(\mathcal{M}) \leq \mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}(\mathcal{M})$.
3. *If $\mathcal{M}$ is $\mathfrak{r}-$exact, then $\mathcal{M}$ is $\mathcal{N}_{\mathfrak{b}}-$exact.*

Proof. The result follows directly from Theorems 4.1 and 4.3. □

Remark 4.4. *Example 4.5 demonstrates that the converse of the previous theorem does not generally hold.*

Example 4.5. *From Example 4.4, we obtain the following right neighborhoods:*

$$\mho_{\mathfrak{r}}(g) = \{g, y, z\}, \qquad \mho_{\mathfrak{r}}(h) = \{g, h, y\},$$
$$\mho_{\mathfrak{r}}(y) = \{y, z\}, \qquad \mho_{\mathfrak{r}}(z) = \{y, z\}.$$

Hence, Table 4 presents a comparative analysis between the prior work, Yao's method, and the $\mathcal{N}_{\mathfrak{b}}-$accuracy (for each $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$).

Table 4: Comparative analysis of accuracy measures between Yao's method and the proposed techniques.

$\mathcal{M} \subseteq \mathbb{U}$	Yao [21] approach	Proposed methods		
	$\mathcal{A}_y(\mathcal{M})$	$\mathcal{A}_{\mathcal{P}_\mathfrak{b}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{S}_\mathfrak{b}}(\mathcal{M})$	$\mathcal{A}_{\gamma_\mathfrak{b}}(\mathcal{M})$
$\{g\}$	0	0	0	0
$\{h\}$	0	1	1	1
$\{y\}$	0	1	0	1
$\{z\}$	0	1	0	1
$\{g,h\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{g,y\}$	0	1	0	1
$\{g,z\}$	0	1	0	1
$\{h,y\}$	0	1	$\frac{1}{4}$	1
$\{h,z\}$	0	1	$\frac{1}{4}$	1
$\{y,z\}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\{g,h,y\}$	0	1	$\frac{1}{4}$	1
$\{g,h,z\}$	0	1	$\frac{1}{4}$	1
$\{g,y,z\}$	$\frac{3}{4}$	1	1	1
$\{h,y,z\}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{U}$	0	1	1	1

(e) Comparative analysis of Abd El-Monsef et al.'s [1] method and the suggested methods

This section presents a comparison between the technique proposed by Abd El-Monsef et al. [1] and the suggested approach. The validated results, supported by a descriptive example, demonstrate that the proposed strategy exhibits superior performance compared to the methods of Abd El-Monsef et al.

Theorem 4.5. *Let $\mathcal{R}$ be a reflexive relation defined on a universal set $\mathbb{U}$, and let $\mathcal{T}_\mathfrak{r}$ denote the topology induced by $\mathcal{R}$. For every subset $\mathcal{M} \subseteq \mathbb{U}$, the following hold:*

1. $\underline{\mathcal{T}}_\mathfrak{r}(\mathcal{M}) \subseteq \mathcal{L}_\mathfrak{b}(\mathcal{M}) \subseteq \mathcal{M} \subseteq \mathcal{U}_\mathfrak{b}(\mathcal{M}) \subseteq \overline{\mathcal{T}}_\mathfrak{r}(\mathcal{M})$.
2. $\mathfrak{B}_\mathfrak{b}(\mathcal{M}) \subseteq Bnd_{\mathcal{T}_\mathfrak{r}}(\mathcal{M})$, *and* $\mathcal{A}_{\mathcal{T}_\mathfrak{r}}(\mathcal{M}) \leq \mathcal{A}_\mathfrak{b}(\mathcal{M})$.
3. *If $\mathcal{M}$ is $\mathcal{T}_\mathfrak{r}-$exact, then $\mathcal{M}$ is also $\mathfrak{b}-$exact.*

Proof. The result follows directly from Theorems 4.3 and 4.4. □

Remark 4.5. *Example 4.6 shows that the converse of the above theorem does not generally hold.*

Example 4.6. *Extending Example 4.5, we construct the topology $\mathcal{T}_\mathfrak{r}$ induced by the relation $\mathcal{R}$ on the universal set $\mathbb{U} = \{\mathsf{t}, \mathsf{u}, \mathsf{v}, \mathsf{w}\}$, together with the family $\mathcal{C}_\mathfrak{r}$ of closed sets defined as,*

$$\mathcal{T}_\mathfrak{r} = \Big\{\mathbb{U},\ \emptyset,\ \{y,z\},\ \{g,y,z\}\Big\} \quad \text{and} \quad \mathcal{C}_\mathfrak{r} = \Big\{\mathbb{U},\ \emptyset,\ \{h\},\ \{g,h\}\Big\}.$$

We then evaluate the $\mathcal{N}_{\mathfrak{b}}-$accuracy measures (for $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$) and compare them with the methods of Abd El-Monsef et al. [1], as reported in Table 5.

Table 5: Comparison of accuracy measures between Abd El-Monsef et al. [1] and the proposed approaches.

$\mathcal{M} \subseteq \mathbb{U}$	Abd El-Monsef et al. [1]	Proposed methods		
	$\mathcal{A}_{\mathcal{T}_{\mathfrak{r}}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{P}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\mathcal{S}_{\mathfrak{b}}}(\mathcal{M})$	$\mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$
$\{g\}$	0	0	0	0
$\{h\}$	0	1	1	1
$\{y\}$	0	1	0	1
$\{z\}$	0	1	0	1
$\{g, h\}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\{g, y\}$	0	1	0	1
$\{t, w\}$	0	1	0	1
$\{h, y\}$	0	1	$\frac{1}{4}$	1
$\{h, z\}$	0	1	$\frac{1}{4}$	1
$\{y, z\}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\{g, h, y\}$	0	1	$\frac{1}{4}$	1
$\{g, h, z\}$	0	1	$\frac{1}{4}$	1
$\{g, y, z\}$	$\frac{3}{4}$	1	1	1
$\{h, y, z\}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$\mathbb{U}$	1	1	1	1

Concluding remarks

The comparative analysis (Tables 2–5) yields the following key findings;

- ✠ The proposed nearly basic-rough set methods demonstrate superior accuracy, achieving values up to 100%, substantially outperforming earlier approaches reported in [5, 8], and [21].
- ✠ Their application enhances the estimation of COVID-19 infections, thereby improving diagnostic reliability.
- ✠ The models are designed to be user-friendly and grounded in topological principles, ensuring accessibility even for non-specialists.
- ✠ The proposed algorithm is readily implementable in MATLAB, offering a practical tool for accurate COVID-19 diagnosis and variant identification.
- ✠ Beyond COVID-19, the framework is versatile and can be adapted to various domains, including other infectious diseases and broader medical contexts (see Section 5).

5 Enhancing Precise Decision-Making in Distinguishing COVID-19 Variants Using $\mathcal{N}_{\mathfrak{b}}-$Rough Set Analysis

By the end of 2019 and the beginning of 2020, the COVID-19 pandemic emerged as a highly virulent virus, characterized by its rapid transmission among humans and its potentially fatal consequences. This outbreak resulted in global crises and disruptions across various aspects of life. Although several vaccines were developed against the original strain, the continuous evolution of the virus and the emergence of multiple variants have raised worldwide concerns, as these mutations are often more severe and transmissible.

The pandemic has also had a profound impact on mental health globally, which in turn has significantly affected the economies of many countries. The primary strains of the coronavirus include three major types: Alpha, Delta, and Omicron. Each of these variants exhibits both common symptoms and variant-specific symptoms. Consequently, the accurate diagnosis of each variant enables physicians to determine the most suitable vaccine and treatment strategy, thereby mitigating the spread of the virus and reducing the likelihood of further rapid mutations.

Motivated by this need, this section introduces a mathematical methodology based on the framework of nearly basic-approximations. The proposed approach aims to assist physicians in making precise decisions regarding the diagnosis of COVID-19 variants. By leveraging the theoretical foundation of $\mathcal{N}_{\mathfrak{b}}-$rough set analysis, this methodology facilitates more accurate decision-making, uncovering hidden data patterns, and saving both time and resources for physicians and patients alike.

These methods are applied to an existing medical information system (see [4, 18]), where the results confirm that the proposed analysis achieves perfect alignment with physicians' diagnostic outcomes, with an accuracy rate of up to 100%. This demonstrates the practical effectiveness of the approach in analyzing the data presented in Table 6.

Furthermore, comparisons with previously established methods cited in the reference list show that the proposed approach outperforms earlier techniques, offering enhanced accuracy and reliability in distinguishing between COVID-19 variants.

5.1 Results from experiments and decision table by doctors

In this section, we present the experimental findings from a preliminary investigation involving an initial cohort of ten patients (see Table 6).

Our study relied on existing medical data originally reported in [4, 18]. Table 6 summarizes the findings related to eight symptoms associated with the three major variants of the coronavirus (Alpha, Delta, and Omicron). The specific symptoms defining each variant are as follows:

(1) $\mathbb{FE}$, $\mathbb{SB}$, $\mathbb{BP}$, $\mathbb{DC}$, $\mathbb{HE}$, $\mathbb{ST}$, $\mathbb{CP}$ for the variant **Alpha**.

(2) $\mathbb{LOT}$, $\mathbb{CP}$, $\mathbb{FE}$, $\mathbb{LOS}$, $\mathbb{HE}$, $\mathbb{SB}$, $\mathbb{ST}$, $\mathbb{CO}$, $\mathbb{MY}$, $\mathbb{FA}$, $\mathbb{BP}$, $\mathbb{RH}$ for the variant **Delta**.

(3) $\mathbb{BA}$, $\mathbb{HE}$, $\mathbb{FA}$, $\mathbb{CO}$, $\mathbb{FE}$, $\mathbb{CL}$, $\mathbb{WE}$, $\mathbb{NS}$, $\mathbb{LBP}$, $\mathbb{LOA}$, $\mathbb{SN}$ for the variant **Omicron**.

Notation for symptoms: $\mathbb{FE}$ (Fever), $\mathbb{HE}$ (Headache), $\mathbb{CO}$ (Cough), $\mathbb{CL}$ (Cold), $\mathbb{SB}$ (Shortness of Breath), $\mathbb{WE}$ (Weakness), $\mathbb{FA}$ (Fatigue), $\mathbb{NS}$ (Night Sweats), $\mathbb{SN}$ (Sneezing), $\mathbb{DC}$ (Dry Cough), $\mathbb{BP}$

(Body Pain), $\mathbb{ST}$ (Sore Throat), $\mathbb{CP}$ (Chest Pain), $\mathbb{MY}$ (Myalgias), $\mathbb{BA}$ (Body Ache), $\mathbb{RH}$ (Rhinorrhea), $\mathbb{LOA}$ (Loss of Appetite), $\mathbb{LOS}$ (Loss of Smell), $\mathbb{LOT}$ (Loss of Taste), and $\mathbb{LBP}$ (Lower Back Pain).

Table 6 reports the diagnoses of patients with Alpha, Delta, or Omicron variants of COVID-19. It is evident that the Omicron variant exhibits several symptoms in common with Alpha and Delta, including $\mathbb{HE}$, $\mathbb{CO}$, $\mathbb{FA}$, $\mathbb{FE}$, $\mathbb{ST}$, $\mathbb{SB}$, $\mathbb{LOS}$, and $\mathbb{LOT}$. However, Omicron is also associated with a distinct set of symptoms ($\mathbb{LOA}$, $\mathbb{LBP}$, $\mathbb{SN}$, $\mathbb{NS}$, and $\mathbb{BA}$) that are less prevalent in the other variants.

Table 6: Dataset of patients and symptoms associated with COVID-19 variants [18, 4].

Person	Symptoms								COVID-19 Decision
	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{a}_6$	$\mathfrak{a}_7$	$\mathfrak{a}_8$	
p_1	$\mathcal{H}$	□	■	■	□	■	□	■	Alpha
p_2	$\mathcal{H}$	□	■	■	■	■	□	■	Omicron
p_3	$\mathcal{H}$	□	□	■	□	□	□	□	Not infected
p_4	$\mathcal{N}$	□	□	□	□	□	■	□	Not infected
p_5	$\mathcal{N}$	■	□	■	□	■	□	■	Delta
p_6	$\mathcal{H}$	■	□	■	□	□	□	□	Delta
p_7	$\mathcal{H}$	■	■	■	□	□	□	□	Delta
p_8	$\mathcal{N}$	□	□	■	□	□	□	□	Not infected

Note: $\mathcal{H}$ = High, $\mathcal{N}$ = Normal, ■ = Yes, □ = No.

5.2 Application of nearly $\mathfrak{b}-$rough set methods in decision-making regarding COVID-19 variants

To illustrate the proposed methodology, consider the dataset in Table 6, where the conditional attributes of symptoms are denoted by,

$$\mathfrak{C} = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8\},$$

and the universe of patients is

$$\mathbb{U} = \{p_1, p_2, p_3, \ldots, p_8\}.$$

The first step is to transform the attributes into qualitative terms, as indicated in the note below Table 6. Next, we compute the similarity of symptoms between any two patients $p_{\mathfrak{g}}, p_{\mathfrak{h}} \in \mathbb{U}$ using the following measure,

$$\Upsilon(p_{\mathfrak{g}}, p_{\mathfrak{h}}) = \frac{\sum_{\ell=1}^{n} [\mathfrak{a}_\ell(p_{\mathfrak{g}}) = \mathfrak{a}_\ell(p_{\mathfrak{h}})]}{n},$$

where $n = |\mathfrak{C}|$ is the number of conditional attributes, and $[\cdot]$ is the indicator function returning 1 when the statement holds and 0 otherwise. This value represents the degree of similarity between two patients.

The resulting similarity matrix is shown in Table 7.

Table 7: Symptom similarities among 8 patients.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
p_1	1	0.875	0.625	0.25	0.625	0.5	0.625	0.5
p_2	0.875	1	0.5	0.125	0.5	0.375	0.5	0.375
p_3	0.625	0.5	1	0.625	0.5	0.875	0.75	0.875
p_4	0.25	0.125	0.625	1	0.375	0.5	0.375	0.75
p_5	0.625	0.5	0.5	0.375	1	0.625	0.5	0.625
p_6	0.5	0.375	0.875	0.5	0.625	1	0.875	0.75
p_7	0.625	0.5	0.75	0.375	0.5	0.875	1	0.625
p_8	0.5	0.375	0.875	0.75	0.625	0.625	0.75	1

From Table 7, the universe of patients $\mathbb{U}$ is classified into two disjoint sets;

- The set of patients afflicted with COVID-19: $\mathcal{Q} = \{p_1, p_2, p_5, p_6, p_7\}$.
- The set of patients unaffected by COVID-19: $\mathcal{W} = \{p_3, p_4, p_8\}$.

Moreover, within the group of COVID-19 infected patients, finer classifications can be made:

- Patients infected with Omicron: $\mathcal{W}_1 = \{p_2\}$.
- Patients infected with Alpha: $\mathcal{W}_2 = \{p_1\}$.
- Patients infected with Delta: $\mathcal{W}_3 = \{p_5, p_6, p_7\}$.

6 Similarity Threshold

In this section, we discuss the role and implications of the similarity threshold within the proposed system. The analysis is presented from both theoretical and practical perspectives.

To satisfy the requirements of the proposed scheme, customized associations are defined. Specifically, the right and basic-right neighborhoods of all patients are constructed, as presented in Table 8. These neighborhoods are based on a relation tailored to the medical context of the problem. Following expert recommendations by physicians, the relation is established as follows,

$$\mathbf{x}\mathcal{R}\mathbf{y} \;\Leftrightarrow\; \Upsilon(\mathbf{x}, \mathbf{y}) \geq 0.75.$$

Mathematical perspective

The threshold value 0.75 denotes the similarity scale; higher magnitudes correspond to greater resemblance, thereby yielding more accurate outcomes. An increase in the similarity ratio enhances decision-making accuracy by enabling more precise classification of elements and fostering strong relationships that aid in identifying variables and accurately classifying patients to distinguish between those infected with the virus and those who are not.

When the similarity ratio approaches unity, Pawlak's method can be applied, providing an exact partition of the approximation space. However, in this case, the binary relation becomes an equivalence relation, which restricts its applicability and prevents the practical use of Pawlak's method.

Therefore, selecting a similarity ratio greater than or equal to 0.75 yields a general relation while maintaining accuracy consistent with Pawlak's results, yet within a broader relational framework that allows greater flexibility, ease of application, and expansion of potential use cases.

It is also important to emphasize that experts can adjust this relation and threshold according to their preferences. Pawlak's classical approach is unsuitable for this system because, although the recommended relation is reflexive and symmetric, it does not satisfy transitivity.

Medical perspective

The similarity threshold plays a crucial role in ensuring the reliability of the decision-making process. In this study, we adopt the value 0.75 as a reference scale, since higher magnitudes of similarity imply stronger resemblance and thus lead to more accurate outcomes. Increasing the similarity ratio improves classification precision, strengthens the relationships among variables, and enhances the ability to correctly distinguish between patients infected with COVID-19 and those who are not.

When the similarity ratio approaches 1, Pawlak's classical rough set method can be applied, yielding an exact partition of the approximation space. However, in this case, the binary relation reduces to an equivalence relation, which limits flexibility and makes Pawlak's approach less practical for real-world medical systems.

By contrast, selecting a threshold greater than or equal to 0.75 produces a more general relational structure. This ensures a balance between accuracy (consistent with Pawlak's results) and applicability within a broader framework, offering flexibility, ease of use, and potential for wider applications in medical decision support.

It is also worth noting that domain experts may adjust both the definition of similarity and the threshold level depending on their requirements. Pawlak's method is unsuitable in this setting because, although the relation is reflexive and symmetric, it does not satisfy transitivity, making it less adaptable to complex medical datasets.

Table 8: Right, maximal, and basic-right neighborhoods.

x	$\mho_{\mathfrak{r}}(x)$	$\mho_{\boldsymbol{m}}(x)$	$\mho_{\mathfrak{b}}(x)$
$p_{\mathbf{1}}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$
$p_{\mathbf{2}}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$
$p_{\mathbf{3}}$	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7\}$
$p_{\mathbf{4}}$	$\{p_4, p_8\}$	$\{p_3, p_4, p_7, p_8\}$	$\{p_4\}$
$p_{\mathbf{5}}$	$\{p_5\}$	$\{p_5\}$	$\{p_5\}$
$p_{\mathbf{6}}$	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7, p_8\}$	$\{p_3, p_6, p_7\}$
$p_{\mathbf{7}}$	$\{p_3, p_6, p_7\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_7\}$
$p_{\mathbf{8}}$	$\{p_3, p_4, p_7, p_8\}$	$\{p_3, p_4, p_6, p_7, p_8\}$	$\{p_8\}$

We now apply the proposed $\gamma_{\mathfrak{b}}-$approximation technique to compute approximations and accuracies for the subsets $\mathcal{Q}$, $\mathcal{W}$, $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$. A comparative analysis is then carried out with results obtained from earlier approaches (Abu-Gadairi [5], Dai et al. [8], and Yao [21]). These comparisons highlight the effectiveness of the proposed methodology in the clinical investigation of COVID-19 variants, as shown in Table 9.

Table 9: Comparative analysis among methods (Abu-Gadairi [5], Dai et al. [8], Yao [21]), and the present approach.

$\mathcal{M}$	**Method of Yao [21]**			**Method of Dai et al. [8]**		
	$\mathcal{L}_{\mathfrak{r}}(\mathcal{M})$	$\mathcal{U}_{\mathfrak{r}}(\mathcal{M})$	$\mathcal{A}_{\mathfrak{r}}(\mathcal{M})$	$\mathcal{L}_m(\mathcal{M})$	$\mathcal{U}_m(\mathcal{M})$	$\mathcal{A}_m(\mathcal{M})$
$\mathcal{Q}$	$\{p_1, p_2, p_5\}$	$\mathbb{U}-\{p_4\}$	42%	$\{p_1, p_2, p_5\}$	$\mathbb{U}$	38%
$\mathcal{W}$	$\{p_4\}$	$\mathbb{U}-\{p_1, p_2, p_5\}$	20%	φ	$\mathbb{U}-\{p_1, p_2, p_5\}$	20%
$\mathcal{W}_1$	φ	$\{p_1, p_2\}$	0	φ	$\{p_1, p_2\}$	0
$\mathcal{W}_2$	φ	$\{p_1, p_2\}$	0	φ	$\{p_1, p_2\}$	0
$\mathcal{W}_3$	$\{p_5\}$	$\mathbb{U}-\{p_1, p_2, p_4\}$	20%	$\{p_5\}$	$\mathbb{U}-\{p_1, p_2\}$	16%
$\mathcal{M}$	**Abu-Gadairi [5] method**			**Proposed technique**		
	$\mathcal{L}_{\mathfrak{b}}(\mathcal{M})$	$\mathcal{U}_{\mathfrak{b}}(\mathcal{M})$	$\mathcal{A}_{\mathfrak{b}}(\mathcal{M})$	$\underline{\gamma}_{\mathfrak{b}}(\mathcal{M})$	$\overline{\gamma}_{\mathfrak{b}}(\mathcal{M})$	$\mathcal{A}_{\gamma_{\mathfrak{b}}}(\mathcal{M})$
$\mathcal{Q}$	$\{p_1, p_2, p_5, p_7\}$	$\mathbb{U}-\{p_4\}$	50%	$\mathcal{Q}$	$\mathbb{U}-\{p_4\}$	70%
$\mathcal{W}$	$\{p_4, p_8\}$	$\{p_3, p_4, p_6, p_8\}$	50%	$\{p_4, p_8\}$	$\mathcal{W}$	67%
$\mathcal{W}_1$	φ	$\{p_1, p_2\}$	0	$\{p_2\}$	$\{p_2\}$	100%
$\mathcal{W}_2$	φ	$\{p_1, p_2\}$	0	$\{p_1\}$	$\{p_1\}$	100%
$\mathcal{W}_3$	$\{p_5, p_7\}$	$\{p_3, p_5, p_6, p_7\}$	50%	$\mathcal{W}_3$	$\{p_3, p_5, p_6, p_7\}$	75%

Discussion

Based on the results presented in Table 9, several important conclusions can be drawn:

- The proposed methods for obtaining high accuracy coefficients demonstrate clear advantages over existing approaches. By employing the nearly basic-approximation techniques, a strong correspondence was established between the medical information in the decision table (Table 6) and the derived mathematical results, achieving accuracy levels of up to 100%. In contrast, previous approaches (Abu-Gdairi [5], Dai et al. [8], and Yao [21]) yielded considerably lower accuracy coefficients compared with $\gamma_{\mathfrak{b}}-$approximations.

- Table 9 further highlights that earlier methodologies encounter significant limitations in diagnosing COVID-19 variants and related epidemics, despite the availability of verified data in the medical decision table.

- The techniques developed in this study enhance precision measures while fully satisfying the theoretical foundations of Pawlak's rough set model, notably without imposing restrictive conditions or assumptions.

- A distinctive strength of the proposed framework lies in its grounding in topological principles, particularly nearly basic-approximations, which inherit several fundamental topological properties. This feature creates opportunities for broader applications of such concepts across diverse domains, including decision-making tasks and extrapolation problems. Moreover, the approach remains accessible even to researchers without extensive backgrounds in topology, thereby expanding its practical relevance and facilitating the design of specialized algorithms with enhanced applicability.

Finally, Algorithm 1 together with the flowchart in Figure 2 illustrates the application of the proposed methods (nearly basic-approximations) to decision-making problems. The algorithm can be implemented using widely accessible programming environments such as MATLAB, thereby enabling the construction of a system capable of producing highly accurate medical diagnoses of COVID-19 and its variants. Furthermore, the same framework can be extended to other decision-making domains, including economic analysis and machine learning applications.

Algorithm 1 Framework for applying near basic-rough sets in decision-making processes.

Input: Construct a table where the set of objects $\mathbb{U}$ is listed in a column (1), and $\mathbb{AT}$, as the set of all attributes, is arranged in a row (1).
Output: Offer an accurate decision for exact and rough sets.

1. **Assess similarity:**
   - Compute the similarity degrees $\varUpsilon\ (\mathfrak{g},\ \mathfrak{h})$ between all attributes for all object by the formula: $\varUpsilon\ (\mathfrak{g},\ \mathfrak{h}) = \dfrac{\sum_{\ell=1}^{n} [\mathfrak{a}_\ell(\mathfrak{g}) = \mathfrak{a}_\ell(\mathfrak{h})]}{n}$, where n denotes a number of Condition attributes.
   - Produce a table showing the similarities between attributes for all objects.
2. **Create binary relation:**
   Provide a binary relation $(\mathfrak{g}, \mathfrak{h}) \in \mathcal{R} \Leftrightarrow \varUpsilon(\mathfrak{g}, \mathfrak{h}) \geq \varepsilon$, such that ε represents the degree of similarity determined based on expert criteria.
3. **Neighborhood computation:**
   For each $\mathfrak{g} \in \mathbb{U}$, make the following:
   - Compute all $\mathfrak{r}-$neighborhoods as defined in Definition 2.2.
   - Determine all $\mathfrak{b}-$neighborhoods as specified in Definition 2.2.
4. **Approximation and accuracy evaluation:**
   For every $\mathcal{M} \subseteq \mathbb{U}$, execute the following:
   Compute the nearly basic-lower approximation $\underline{\mathcal{N}}_{\mathfrak{b}}\ (\mathcal{M}), \forall \mathcal{N} \in \{\mathcal{P},\ \mathcal{S},\ \gamma\}$, using Definition 3.1.
   - If $\underline{\mathcal{N}}_{\mathfrak{b}}\ (\mathcal{M}) = \Phi$, then $\mathcal{M}$ is a rough set.
   - Else, make the next:
     1. Calculate the nearly basic-upper approximation $\overline{\mathcal{N}}_{\mathfrak{b}}\ (\mathcal{M}), \forall \mathcal{N} \in \{\mathcal{P},\ \mathcal{S},\ \gamma\}$, following Definition 3.1.
     2. Determine the nearly basic accuracy $\mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}\ (\mathcal{M}) = \dfrac{|\underline{\mathcal{N}}_{\mathfrak{b}}\ (\mathcal{M})|}{|\overline{\mathcal{N}}_{\mathfrak{b}}\ (\mathcal{M})|}$.
   - If $\mathcal{A}_{\mathcal{N}_{\mathfrak{b}}}\ (\mathcal{M}) = 1$, classify $\mathcal{M}$ as an exact set.
   - Else, $\mathcal{M}$ is a rough set.

End

Figure 2 illustrates a straightforward flowchart depicting the accuracy measures derived from the aforementioned algorithm.

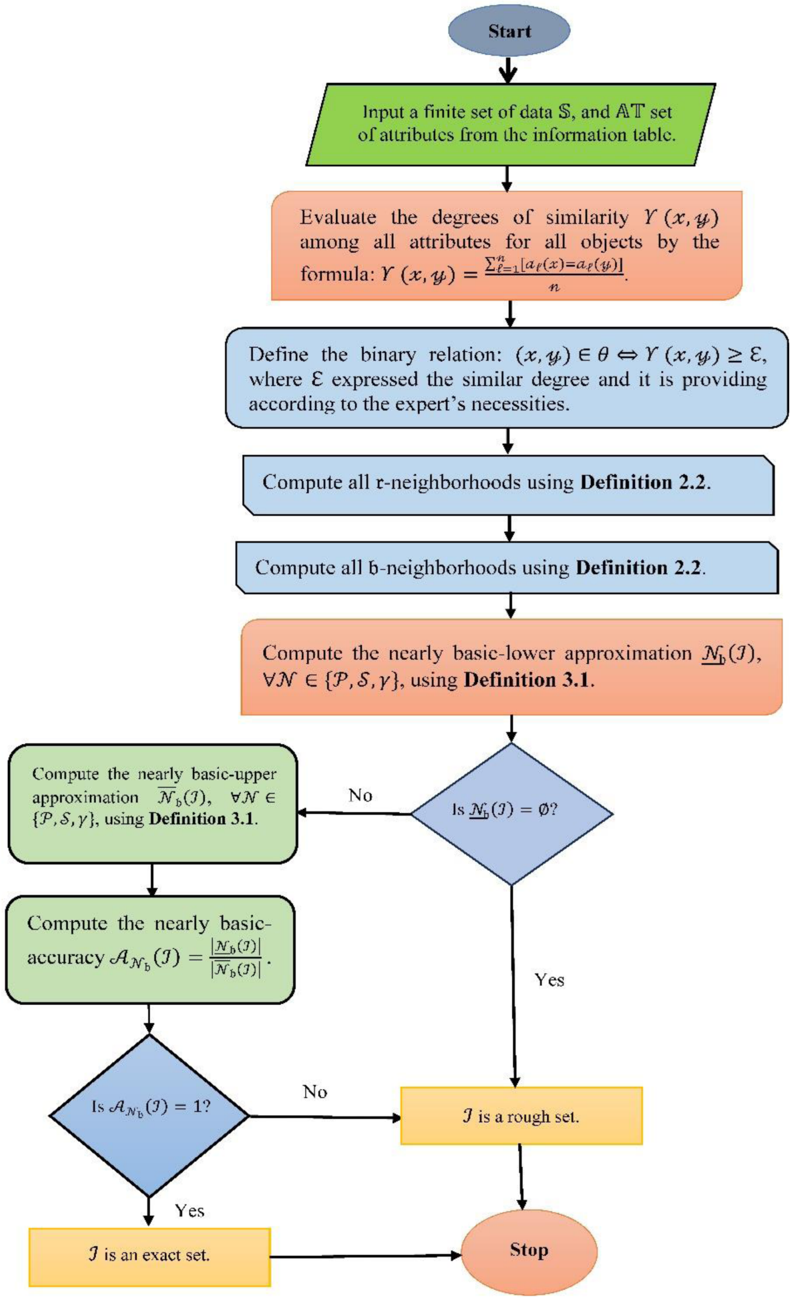

Figure 2: Flowchart illustrating the utilization of near-basic approximations in decision-making challenges.

7 Conclusion and Future Directions

In this study, we introduced and analyzed novel extensions of generalized rough sets, termed nearly basic-approximations ($\mathcal{N}_\mathfrak{b}-$approximations, for every $\mathcal{N} \in \{\mathcal{P}, \mathcal{S}, \gamma\}$), within the framework of topological structures.

These methodologies, grounded in neighborhood systems and accessible even to non-specialists, significantly enhance approximation operators and accuracy measures compared with existing techniques. By extending Pawlak's classical rough set models, they preserve generality while improving applicability to a wide range of practical problems. Comparative analyses with earlier approaches demonstrate superior accuracy, particularly in medical diagnostic tasks such as identifying COVID-19 variants.

Advantages of the proposed approach

1. **Flexible modeling:**
The proposed framework enables flexible modeling of diverse problems while minimizing restrictive assumptions.

2. **Simplified application:**
Direct reliance on graph-based data enhances ease of use, making the approach accessible even to researchers without extensive expertise in topology.

3. **Improved accuracy:**
Broadened approximation operators and refined accuracy measures support more precise decision-making, with particular advantages in medical diagnosis and machine learning.

4. **High performance:**
The methods achieved accuracy rates of up to 100% in diagnosing COVID-19 variants, clearly surpassing previous approaches.

5. **Algorithmic implementation:**
The framework is supported by a MATLAB-compatible algorithm, which facilitates practical deployment and usability.

Future research directions

1. **Broader applications:**
Extend the use of $\mathcal{N}_{\mathfrak{b}}-$approximations to additional domains such as medicine, economics, and engineering, in order to assess their effectiveness across diverse real-world contexts.

2. **Theoretical extensions:**
Investigate the integration of $\mathcal{N}_{\mathfrak{b}}-$approximations with other frameworks, including Rough Fuzzy Sets, Soft Rough Sets, Fuzzy Topological Spaces, and Graph Theory.

3. **Hybrid methodologies:**
Explore the incorporation of probabilistic and likelihood-based techniques to further enhance accuracy and practical effectiveness in complex real-world decision-making scenarios.

Data availability

The data utilized in this study is available in [4, 18].

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Conflicts of Interest The authors declare no conflict of interest.

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